Multifractality of Self-Avoiding Walks on Percolation Clusters

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We consider self-avoiding walks on the backbone of percolation clusters in space dimensions $d = 2, 3, 4$. Applying numerical simulations, we show that the whole multifractal spectrum of singularities emerges in exploring the peculiarities of the model. We obtain estimates for the set of critical exponents that govern scaling laws of higher moments of the distribution of percolation cluster sites visited by self-avoiding walks, in a good correspondence with an appropriately summed field-theoretical $\varepsilon = 6 - d$ expansion [H.-K. Janssen and O. Stenull, Phys. Rev. E **75**, 020801(R) (2007)].

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When studying physical processes on complicated fractal objects, one often encounters the interesting situation of the coexistence of a family of singularities, each associated with a set of different fractal dimensions [1]. In these problems, the conventional scaling approach cannot describe the system. Instead, an infinite set of critical exponents is needed to characterize the different moments of the distribution of observables, which scale independently. These peculiarities are usually referred to as multifractality [2]. The multifractal spectrum can be used to provide information on the subtle geometrical properties of a fractal object, which cannot be fully described by its fractal dimensionality. Indeed, clusters generated by diffusion-limited aggregation [3] and percolation clusters have the same fractal dimensions, but a completely different geometrical structure, which can be clarified, e.g., by studying the multifractality of the voltage distribution in percolation clusters [4] and the growth probability distribution in diffusion-limited aggregation [5,6]. Multifractal properties arise also in many different contexts, for example, in studies of turbulence in chaotic dynamical systems and strange attractors [2,7], human heartbeat dynamics [8], Anderson localization transition [9], etc.

To understand the common roots underlying this phenomenon, it is worthwhile to consider the generic case of self-avoiding walks (SAWs) on fractal clusters. It is well established that configurational properties of SAWs on a regular lattice are governed by scaling laws; e.g., for the averaged end-to-end distance $\langle r \rangle$ of a SAW with N steps one finds in the asymptotic limit $N \to \infty$:

$$\langle r \rangle \sim N^{\nu_{\mathrm{SAW}}}, \tag{1}$$

where ν_{SAW} is a universal exponent depending only on the space dimension d.

The scaling of SAWs changes crucially when the underlying lattice has a fractal structure. Indeed, new critical exponents were found for SAWs residing, e.g., on a Sierpinski gasket and Sierpinski carpet [10]. A related problem arises when studying SAWs on disordered lattices with concentration p of structural defects very close to the percolation threshold p_c. In this case, an incipient cluster of pure sites can be found in the system. The diameter of a typical cluster below p_c is characterized by the correlation length ξ, which diverges as $\xi \sim (p - p_c)^{-\nu_p}$ with a universal exponent ν_p. Note that percolation clusters are fractal objects (see Table I) and apparently change the universality class of residing SAWs; the scaling (1) holds in this case with an exponent $\nu_{p_c} \neq \nu_{\mathrm{SAW}}$. Aiming to study the scaling of SAWs on a percolative lattice, we are interested rather in the backbone of percolation cluster: the structure left when all "dangling ends" are eliminated from the cluster. Infinitely long chains can only exist on the backbone of the cluster.

Although the behavior of SAWs on percolative lattices has served as a subject of numerous numerical [16–21] and analytical [22–25] studies since the early 1980s, not enough attention has been paid to clarifying the multifractality of the problem. Following an early idea of Meir and Harris [17], it was only recently proven in field-theoretical studies [24,25] that the exponent ν_{p_c} alone is not sufficient to completely describe the peculiarities of SAWs on percolation clusters. Instead, a whole spectrum $\nu^{(q)}$ of multifractal exponents emerges [25]:

$$\nu^{(q)} = \frac{1}{2} + \left(\frac{5}{2} - \frac{3}{2^q}\right)\frac{\varepsilon}{42} + \left(\frac{589}{21} - \frac{397}{14 \times 2^q} + \frac{9}{4^q}\right)\left(\frac{\varepsilon}{42}\right)^2, \tag{2}$$

with $\varepsilon = 6 - d$. Note that putting $q = 0$ in (2), we restore an estimate for the dimension $d^B_{p_c}$ of the underlying back-

TABLE I. Fractal dimensions of percolation cluster $d^F_{p_c}$ and backbone of the percolation cluster $d^B_{p_c}$ and correlation length critical exponent ν_p for different space dimensions d.

d	$d^F_{p_c}$	$d^B_{p_c}$	ν_p
2	91/49 [11]	1.650 ± 0.005 [12]	4/3 [13]
3	2.51 ± 0.02 [14]	1.86 ± 0.01 [12]	0.875 ± 0.008 [15]
4	3.05 ± 0.05 [14]	1.95 ± 0.05 [12]	0.69 ± 0.05 [12]

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for $d = 2$ with the exactly known multifractal spectrum for the harmonic measure describing diffusing particles at the surface of percolation clusters [32], we have added the inset in Fig. 4. That this is not a theory for the present system, however, is apparent already from the 0th moment: in our case it gives the "bulk" fractal dimension $d^B_{p_c}$ of the percolation cluster, whereas in the latter case it corresponds to its accessible external perimeter.

It is well known that the set of exponents governing scaling of multifractal moments of the type (4) is related to the spectrum of singularities $f(\alpha)$ of the fractal measure [5], called also the spectral function. The physical meaning of $f(\alpha)$ in our problem is that the number $N_R(\alpha)$ of sites i, where the weight $w(i)$ scales as $R^{-\alpha}$, behaves as

$$N_R(\alpha) \sim R^{f(\alpha)}. \tag{5}$$

The singularity spectrum $f(\alpha)$ is given by the Legendre transform:

$$f(\alpha) = q\alpha - \tau(q), \qquad \alpha(q) = \frac{d\tau(q)}{dq}, \tag{6}$$

with $\tau(q) = 1/\nu^{(q)}$. Spectral functions $f(\alpha)$, obtained on the basis of our results, are given in the lower panel of Fig. 4. The general properties of $f(\alpha)$ are as follows: it is positive on an interval $[\alpha_{\min}, \alpha_{\max}]$, where $\alpha_{\min} = \lim_{q\to+\infty}\tau(q)/(q-1)$, $\alpha_{\max} = \lim_{q\to-\infty}\tau(q)/(q-1)$. The maximum value of the spectral function gives the fractal dimension of the underlying structure, which in our case corresponds to the fractal dimension $d^B_{p_c}$ of the backbone of percolation clusters.

To conclude, we have shown numerically that SAWs residing on the backbone of percolation clusters give rise to a whole spectrum of singularities, thus revealing multifractal properties. To completely describe peculiarities of the model, the multifractal scaling should be taken into account. We have found estimates for the exponents, governing different moments of the weight distribution, which scale independently, in surprisingly good coincidence with two-loop ε expansions. The behavior of the spectral function, describing the frequency of observation of a set of singularities on the underlying backbone of percolation clusters, is analyzed as well.

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